Mason, J. 2002, Exploiting Mental Imagery in Teaching & Learning Mathematics, *Actas ProfMat 2002* Conf 13 p75-81.

Translated as

Mason, J. 2002, (S. Diego Trans.) Explorando Imagens Mentais no Ensino – Aprendizagem da Matemática, *Actas ProfMat 2002* Conf 13 p82-91.

EXPLOITING MENTAL IMAGERY IN TEACHING & LEARNING MATHEMATICS

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ABSTRACT

I use the term *mental imagery* to refer to anything and everything that happens inside you when you are thinking, planning, considering, and reflecting. For some it may be predominantly visual, for others predominantly verbal, and for others predominantly visceral; for most there will be elements of all of these.

The power to imagine is perhaps the most important and fundamental of all the many powers which children possess when they come to school, indeed when they emerge from the womb. In particular, it forms a world which lies between the outer world of material objects, and the outer world of abstract symbols. The issue here is how we can exploit that power to the full. My aim is to illustrate different ways in which imagery and imagination can be used effectively in the mathematics classroom and in preparation for lessons.

I will invite participants to explore this power in themselves, and use that experience to draw out various ways in which that power can be exploited not only in the mathematics classroom, but in advance when preparing and afterwards when reflecting. If there is time, I will go so far as to suggest that professional development and researching your own practice are based upon use of this power.

INTRODUCTION

Methods

My method of enquiry is to identify phenomena I wish to study and to seek examples within my own experience. I then construct task-exercises to offer to others, in order to see if they recognise what I find myself noticing. Through refinement and adjustment of task-exercises in the light of experience and of reading relevant literature, I both extend my own awareness, and offer experiences to others which may highlight or even awaken sensitivities and awarenesses for them. These sensitivities and awarenesses can in turn inform our future practice. As task-exercises are developed and shared, actions or tactics which exploit what is noticed for the benefit of students, are incorporated into teaching.

For example, given that I want to study the role of mental imagery in the teaching and learning of mathematics, it is vital that I revisit aspects of mental imagery for myself, with the assumption that this re-visiting will be important for you as well. It enables me to speak from recent experience, and it enables us to obtain some measure of agreement as to what we may be talking about. Consequently, I shall begin with some task-exercises for you the reader, followed by comments in which I draw on some of my experiences of working on those tasks. My conjecture is that working on these tasks will trigger recognition of similar situations you have experienced in the past, and that my comments will then make sense because they fit with, contrast with, or illuminate that experience. My method does not attempt to capture or address every experience of every reader. Rather it aims to make contact with that experience, perhaps challenging interpretations, perhaps pointing to features not previously noticed.

The data of my method are the experiences generated, the sensitivities to notice which are enhanced. If you recognise at least something of what I am talking about as a result of having worked on these tasks, you may be stimulated to look out for similar experiences in the future, and over time, begin to act upon what you notice. Validity in this method is personal. Validity for you lies in your finding your actions being informed in the future, not in what I or others say, and not in what it is claimed that others experienced.

A consequence of my way of working is that others must participate in tasks if they are to appreciate anything of what I am trying to point to. For what is important lies beyond words. It lies in the world of mental imagery which is accessed through actively noticing and questioning what is noticed in the way of how you use your powers, of how you use yourself (see Mason 2002 for elaboration).

First Steps

I use the term imagery very broadly to refer to all inner experiences, including inner pictures, inner sensations, inner sounds, inner-positioning, inner-posturing, a fuzzy sense-of, and so on. It is what happens inside us when we receive sense-impressions, and when we experience inner virtual sense-impressions, whether 'remembered', eidetic, or intentionally constructed. For example:

> I am going to clap a few times. Your task is to count how many claps I make.

Comment

If I do several claps according to some rhythm so that you cannot count them as I clap, you find nevertheless that you can instantly replay the rhythm internally and perform the count.

This is just one aspect of imagery: the power to replay inwardly what has recently happened outwardly through virtual sense impressions. More broadly,

> Imagine going into your kitchen to get a spoon. On which side of the sink is the cutlery to be found?
>
> Imagine ... a classroom in which you are going to teach on Monday, or a room in which you will work on Monday. Where will you be standing? Who is on the edges of your vision as you look at the class as a whole? Try imagining standing somewhere else in the room.

Your response is likely to bring to the surface physical sensations, emotional states, and cognitive connections. All is subsumed under imagery. See how precise you can be, how detailed. You may not be able to 'see' detail, but you can increase the intensity of the sense of 'being there' in the room simply by exercising your will to do so. Once, while driving in the car, I asked my son aged about 5 if he could re-enter his bedroom in his mind. He went very quiet, and about five minutes later simply said "yes"! On enquiry it turned out he had been working at very fine detail!

What has this to do with mathematics?

> Imagine a triangle. Move it about in the plane, rotating, translating, changing size. Get a sense of the freedom available. Now fix it in your mental plane. Imagine also a straight (infinite) line in the same plane. Let the line move about so that you get a sense of the freedom available to it, and of the ways in which it can interact with the triangle. It could be parallel to an edge, pass through some special points like vertices,

mid-points of edges, and so on. Could it cut all three edges (extended) at the same angle?

Whatever you do in response, you are using your powers of mental imagery. Language can evoke activity in the mind other than straight processing of words, and that too is mental imagery. Direct, imperative language is much more effective than any other form.

Rather than provide a definition of imagery, I find it more fruitful to examine with an open mind the wide range of situations in which the word *imagery* is or has been used.

An image can be

as clear as your awareness of which side of your sink the cutlery is to be found;

as transient as a fleeting sense of recognition as someone walks past you;

as vivid as a scene which often comes to mind, such as a particular place, or a particular room;

as detailed as a reconstructible generic 'picture of' an instance of a generality, from or through which the general theorem can be read;

as ephemeral as a sense-of potential generality;

as multiple as a song which won't go away but which comes with an image of a favourite place, while you simultaneously consider what you will have to eat from a menu.

as indefinite as your sense of yourself and where you are going in life: like a beacon on a hill that is often obscured by trees and hills, providing direction for actions but remaining illusive and out of reach; to speak about it is to tumble it into substantiality like images of chairs and lemons and triangles and thereby turn it into something else;

Images rapidly become abstracted. The word 'sunset' may not trigger any particular sunset, but more a sense of sunset-ness. If you are told that you are going to the cinema, you suddenly have a sense of cinema-ness. The language of frames (Minsky 1975) and the language of scripts (Schank & Abelson 1977) provide two ways of speaking about image-based expectations which are triggered. If asked to describe 'a cinema', you are likely to access part of your cinema-frame or a cinema-going-script. You might work from a particularly vivid image of a specific cinema, or you might find yourself working from a generalised or abstracted image, perhaps composed of fragments of images of actual cinemas. As mathematics teachers, we would presumably like students to have access to similarly abstracted or generalised images rather than being confined to images of particular triangles or particular functions. In other words, we want our students to look *through the particular to the general*. Mental imagery is where and how that is done.

Complexity and Power of Mental Imagery

Because I am encompassing all inner experiences which are sensation-like, as well as a more fuzzy and inchoate 'sense-of', perhaps linked to intuition, mental imagery is both powerful and complex:

Imagine entering your kitchen and getting a spoon; while you are doing that, recall and 'hear' a favourite tune; in addition, recite mentally the counting poem one-two-three…; (notice whether the tune and counting interfere);

Comment

If you had trouble with some of that, be assured that you can do all this and more with a little bit of practice. Robertson (2002) offers a splendid account of how to do this.

Imagery is the power behind learning to use language, for learning language is primarily about learning to express yourself using words. When recounting a story at home about what happened at school, young children famously act as if their audience can 'see what they see' in their mind. Indeed, young children dwell in imagery more than in words (Robertson 2002 p23-24). Part of the process of getting them to write fluently is awakening them to the fact that others have no access to their minds. Through awareness of their own mental imagery, they learn to write *from* that imagery, rather than to talk about it or as if it were shared.

As well as being the source of expressive language, it is through becoming aware of our mental imagery that we become expressive, articulate, and communicative. The same applies to mathematics. It is through entering and using mental imagery that we encounter the mathematics that lies beyond speech and symbols, and it is through struggling to express those inchoate intuitions and awarenesses that we turn experience into formal mathematics.

Somehow, in the process of abstracting numbers as nouns from numbers as adjectives for collections of objects, many children pick up the idea that mathematics is something which just happens, or fails to happen; that symbols flow out of pens attached to arms of people who are clever. This is so far from the truth! Mathematics is something you engage in, through your mental imagery. You perform actions, whether physical, mental, or symbolic. You sense or detect relationships. You express these relationships as conjectures, and you try to 'see' why your conjectures are correct or where they need modifying. The real work takes place internally; the outer behaviour is only a small part of the whole. But it is all or most of what we have to go by! Dorfler (1991) offers a way of thinking about how mathematical understanding develops, based on image schema, which in turn depend on our power to form mental images.

Mathematics also involves disengagement from immediate action. It requires standing back from doing calculations in order to get a sense of *how* the calculations are done, in order to generalise the method as a technique, and to appreciate the dimensions-of-possible-variation (Marton & Booth 1997, developed in Mason 2002a) which constitute that 'type' of problem. Classes of problems are resolved by extracting an essence through ignoring certain details, in order to see through particulars to a generality. In order to see through particulars it is vital to be able to enter the world of the problem and to experience it virtually, in order to discern what is relevant and to detect and express relationships.

The place of imagery at the heart of mathematical experience is only a particular instance of the central role of imagery in human experience generally. In his introduction to Shaffer's play *The Gift of the Gorgon*, (Shaffer 1993), Peter Hall wrote

> Shaffer has used the cinematic sophistication of his audience to reinforce an old strength of the theatre: its ability to invite an audience to imagine. Paradoxically this is something the cinema can never do: the images remain what they are - literal. It is for the theatre to be a place of metaphors.

Perhaps Shaffer had in mind Shakespeare's prologue to Henry V:

> Think, when we talk of horses, that you see them,
> Printing their proud hoofs i'the receiving earth.
> For 'tis your thoughts that now must deck our kings,
> Carry them here and there, jumping o'er times;
> Turning accomplishment of many years
> Into an hourglass.

Someone once remarked "I prefer radio to television because the pictures are much better"! It seems then that imagery is a fundamental power possessed by all human beings. It is at the heart of expressiveness when listening to or reading the expressions of others; it is the core and root of expressing oneself to others, and hence of making sense for oneself. It is therefore available to be exploited in doing, learning, and hence in teaching mathematics.

USING IMAGERY TO DO MATHEMATICS

This and the following sections are but brief indications of a rich world of exploration and activity in which imagery can be exploited. Dreyfus (1991) stresses the role which imagery plays for mathematicians, and most significantly, perhaps, the ways in which imagery is often obscured or suppressed by the published formalisms, requiring other mathematicians to try to reconstruct the diagrams and images hidden by the symbols. Interestingly, there has been a rise in the use of diagrams to condense complex symbolic expressions so as to enable mathematicians to think through and around those computations: Feynman and Coxeter diagrams are just two of several diagram systems.

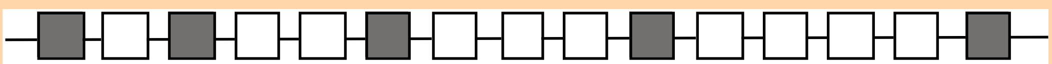
SIMPLE BEGINNINGS

I began with counting claps because it so clearly calls upon the power to form mental images. Here are some developments of clapping, aimed at young children.

At the same time as clapping hard for a grey and soft for a white, count out loud 1, 2, 3, 4, … using the following sequence.

grey, white, grey, white, white, grey, white, white, white, grey, white, white, white, white, grey, …

Alternatively it may help to use a visual display to get started, but continue well beyond the ones shown here! These are meant to suggest beads on a string.



Comment

Most people find that quite hard the first time. You have to relax, and become aware that at the same time as counting out loud you can mentally count the next number of soft-claps (white squares) before the next strong clap. There is a moment of transition from not-succeeding to succeeding, which signals the use of one aspect of the power to imagine. You may want to develop confidence by resorting to a simper task (a core mathematical move in the face of complexity) but it is well worth persevering in order to become aware of the power of mental imagery. The desire to specialize, to try something simpler is excellent mathematics, as long as the point is to try to get a sense-of what is going on so as to inform work on the more complex problem.

Mental images are not just sensations-in-the-mind. They are the means by which we guide and direct our behaviour, overlaid on automatic physical movements like clapping.

If you experienced interference between the clapping and the counting, you have a taste of the sort of interference that young children may experience when they are learning to do simpler clapping rhythms, or learning tables. When their attention wanders, they lose the beat. It may also be analogous to what happens when they are trying hard to do one task which involves a sub-task that also requires a lot of attention. Thus they may appear to 'fail' to do something that you 'know they can do'. When attention is diverted or distracted, competencies may become inaccessible.

Turning from using a diagram as a memory aid to diagram as source of patterns:

Look at the chart on the left for a few seconds. Then cover it up, close your eyes and say to yourself what you saw, then try to reproduce it. Then make a list of all the features that you notice about it.

				. . .				
	-9	-6	-3	0	3	6	9	
	-6	-4	-2	0	2	4	6	
	-3	-2	-1	0	1	2	3	
. . .	0	0	0	0	0	0	0	. . .
	3	6	9	0	-1	-2	-3	
	2	4	6	0	-2	-4	-6	
	1	2	3	0	-3	-6	-9	
				. . .				

Comment

Notice the stabilising effect of having something to look at. A diagram or chart is not in fact static. Or rather, what you can do with a chart or diagram is impose extra elements, imagine some elements changing, and so on. There is a curious dynamic which you can bring to something seen, by focusing and diffusing your attention, and moving it around. This is much harder with music and speech which are manifested in time. With the bead-string, the images are for many people easier to use as a prompt to assist the clapping than are the words. In the chart, I hope you were aware of patterns continuing in all directions, and that you became aware that the simple 1, 2, 3, sequence could be replaced by other sequences.

Say What You See is a useful device to use with physical situations, electronic screens, mathematical posters, diagrams, sets of exercises, and even single exercises. By getting people to describe without pointing, they learn to express themselves clearly and concisely. By hearing what others 'say they see', they discover that working together can be richer than working alone, and that there may be other more fruitful things to discern. For what we discern through our imagery is the basis for the sense that we make. Even with a single expression it is sometimes amazing what people apparently do not see!

Say what you see here: $\frac{\sqrt{7} \square \sqrt{5}}{\sqrt{7} + \sqrt{5}} \square \frac{\sqrt{5} \square \sqrt{3}}{\sqrt{5} + \sqrt{3}}$

Comment

Some people are put off by the square-roots and see only a mess; others see shape and pattern and can re-construct it easily.

I am convinced that to get much educational benefit, students need to be active in processing images; they need to *work on* the images, not just *look at* them. They need to probe beneath surface reactions. Working on and with mental imagery supports this development.

ALGEBRA AND GEOMETRY USE IMAGERY

Algebra is the study of the dynamics of relationships (Gattegno 1987). These relationships are conceived and experienced internally through a mixture of visual, aural, visceral and holistic senses. Gattegno also suggested that geometry is the study of the dynamics of the mind, of mental imagery. The diagrams that are drawn in books, the diagrams drawn on electronic screens, are manifestations of particulars. But the mind encompasses generality. Unfortunately exposure to pictures on screens emphasizes the particular rather than the general, which is why in dynamic geometry it is useful to be able to move objects around. As Shaffer indicated, physical screens struggle to indicate generality.

Imagine a square. Move it around (everything stays on your mental plane!). it can rotate, and translate; it can change size as it moves. But it always remains a square. Get a sense of the freedom possessed by your square. Now fix your square so you can be aware of all of it. Introduce a second square. Get a sense of the freedom of your second square in relation to the first, changing size, rotating, translating. Whenever your second square overlaps the first, let the overlap region be a different colour. Now consider all the different shapes that you can form as your second square overlaps the first.

Comment

A static image like the multiplication chart can stimulate mental dynamics 'on top'; dynamic images on screens may give access to generality, but may inhibit the use of personal imagery.

The task has produced a classification problem. Learners have to decide what counts as the same and what as different, a type of decision which underlies most of human experience and certainly a great deal of mathematics. Instead of working with shapes like squares we can work with points, lines, and circles, and gain access to all the fundamental awarenesses which are formalised in Euclidean geometry. Due to lack of space I can only refer you to the influential and inspiring book *Geometric Images* (Leapfrogs 1982).

It isn't necessary to be able to see all the detail in an image, indeed one of the strengths of mental imagery is the way you can focus and de-focus, and get a sense-of without being plagued by details.

Imagine a regular chiliagon (a 1000-sided regular polygon). Now imagine a regular 999-gon. Imagine all the chords drawn in. What is the difference between the two images?

Comment

It is impossible to 'see' all the chords, even if you actually look at a picture! But it is still possible to imagine them all being present and available. Dennett (1991, p297) observes that for Descartes it was possible to *conceive* of a chiliagon and of a 999-gon, but that it was not possible to *imagine* them differently, and in this way he distinguished between *conception* and *imagination*. Certainly if you display them both on a human sized screen it would be hard to distinguish between them! Dennett goes on to suggest that 'to imagine' is an act that can be accomplished, can be embarked upon deliberately, whereas 'to conceive of' is not.

I find it rather difficult to sustain Descartes' distinction, for my conception of the chiliagon and the 999-gon have much more substance than just what I can actually see. I have access to the number of sides as well, and hence to a whole range of properties and calculations, even though I could not identify the difference between material pictures of the two. For example, my images both have a single vertex at the top, but one has one at the bottom and the other has two; one has a lot of chords passing through the centre, the other has none. Furthermore, I can work deliberately at juxtaposing the visual, aural, and other sense-based attributes which with my 'knowledge of' and 'knowledge about', make up the totality of my image. Tall and Vinner (1981) introduced the term *concept image* to refer to this totality of connections and associations, and the idea is developed in Tall (1991).

Imagined and constructed generalised images sit side by side sensory images of mathematical apparatus and diagrams. Lacan (1985) links imagination not based on sensation with the symbolic:

> '...there is nothing in the nature of the wheel that will describe the pattern of marks that any one of its points makes on each turn. There is no cycloid in the imaginary. The cycloid is a discovery of the symbolic' (p208)

I see the cycloid as lying entirely in the imagination. It is spawned or released, by exercise of the symbolic through equations and graphing, but the cycloid itself is conceived-imagined. It can even be experienced: seven or eight people hold hands in a circle can 'roll' their circle along a straight line. Participants soon get a sense of what it is like to be a point on a wheel, and what motion along the cycloid is like.

Modelling & Problem Solving

Mental Imagery constitutes a world of experience which lies between the material or physical world, and the world of symbols in which algebra and geometry are performed. For example, mathematical modelling starts from recognizing something as problematic, as unexpected and needing explanation or prediction, usually in the material world. The problematicity sets up mental imagery in which different features are imagined, considered, and evaluated for pertinence to the problem. This is where modelling assumptions are made, where simplification takes place, and where identification of relevant variables and relationships happens. Those relationships are then expressed diagrammatically and ultimately, usually, in symbols. In the symbolic world the problem is reposed and expressed within formal mathematics, and if possible resolved. The resolution is then reinterpreted using the diagrams and insights from the mental imagery, and inserted back into the original source situation. If the original problem is not fully resolved, more cycles of refinement are undertaken, as necessary.

My only reason for rehearsing this view of modelling (Open University 1971, Mason 2001) is to draw attention to the crucial and critical work which takes place in the world of mental imagery. Of course modelling is a special case of problem solving, which likewise involves movement between worlds, and in which mental imagery also plays a vital role. As a brief example:

Imagine a plastic cup rolling about on the floor. What is its path?

Comment

You immediately imagine the rolling extended, and you can sense (perhaps see, perhaps intuit) what shape it describes. When you ask yourself what the radius of the circle will be in terms of the dimensions of the cup, you are already in the world of mental imagery where you imagine or draw a diagram, label relevant parts, and recognize relationships. Expressing these relationships in symbols allows you to compute the answer.

You might focus on the can from which the drink comes, and discover that sometimes it balances on the lip on the bottom (Mason 2001)! You immediately find yourself imagining the liquid inside, the shapes, the forces, …. Again you are poised on the edge of modelling, making use of mental imagery, perhaps even tumbled into working on the problem through the arising of mental images. For it is by means of mental images that we exercise direction in moment-by-moment actions, and by means of mental imagery that we are inspired and moved to engage with the material world.

Imagery, Anticipation, and Symbol Manipulation

Pauolo Boero (1996) has highlighted a feature of symbol manipulation which involves mental imagery. Mathematicians do not just fiddle with symbols, randomly combining them in different ways. They manipulate symbols towards a desired end, for a purpose. And furthermore, they anticipate the result of calculations, sometimes in detail, sometimes in form. This anticipation is a functioning of the power of mental imagery.

Before embarking on a course of action trying to resolve a problem or problematic situation, it is helpful to ask yourself "what would a resolution look like?". In non-academic situations this tends to happen naturally, as can be verified by noticing what happens when someone proposes a resolution or a course of action which does not fit with your expectation. Expectations are often lying below the surface, coming to the fore only when they are disturbed in some way. Expectation is the anticipatory function of mental imagery: broken expectation creates surprise-disturbance; disturbance activates sense-making. If you ask people about their expectations for a session at a conference, most find it hard to be articulate; but once the session has started, they can tell you quite quickly whether this is what they were expecting! Yet when they do a mathematics problem, the same people often have quite a strong sense of what a reasonable answer might look like. Part of the role of a teacher is to assist learners in developing their mathematical anticipation, their mathematical expectations.

Further evidence of the role of anticipation can be experienced through the interventions of an inner monitor: the part of you which suddenly asks "Why am I doing this?", "Does this look right?", "Is it meant to be this complicated?", and so on. Schoenfeld (1985) refers to an inner executive, while Mason, Burton & Stacey (1982) call it an 'inner monitor'. Mental imagery is the means by which anticipation is used to direct activity. More generally, mental imagery is the means whereby we guide and direct our long-term activity, which then informs our moment-by-moment choices. An inner monitor can be strengthened and encouraged by developing control over mental imagery, and by extending the range of types of images to which you have access. Like any power, mental imagery can be developed and enhanced.

USING IMAGERY TO LEARN MATHEMATICS

I have recently encountered some students who do not know how to study mathematics for an examination. Indeed I have heard some say "I've done all the tasks I was set … what else is there to do?". I suggest that mental imagery can play a significant role in preparing for an examination, as well as in the doing of mathematics.

> Add 972 and 784; divide 8637 by 48; find the greatest common divisor of 1364 and 7396; find my present age if 7 years ago I had worked at the Open University for half a year more than half my life, and in 4 years time I will have worked at the Open University for one-fifth year more than 3/5ths of my life.

Comment

There are methods for doing each of these: the methods are general, applicable to 'any such question'. The important thing about being set tasks like these is not to get the answers. The purpose is to become aware of the generality of various possible methods: when they work, when they are efficient, when they need modifying. Building on the notion of dimensions-of-variation (Marton & Booth 1997) as the basis for learning, my wife Anne Watson and I have augmented it to the dimensions-of-possible-variation in a task, which constitute our sense of its class or type, and also constitute our awareness of generality. With each dimension there is a range-of permissible change. Learning make involved extending this range, or adding new dimensions. Thus you can add any number of multi-digit numbers even though you might be hard pressed to articulate the details of exactly how in all possible cases: you are aware of the number of digits, the range of possible digits, and perhaps even the possibility of changing bases; you can divide any two numbers, but again, articulating the details in general is virtually impossible (try building a spreadsheet to do digit-by-digit arithmetic operations!), and you can work out my age by guess-&-test, by try-&-improve, or by spot-&-check, or by using the checking of a guess as a template from which to write down equations using a symbol for the as-yet-unknown, and so on. But you can do the same for other variations of this task, opening access to a whole genre of age-tasks stated as word problems and popular since the middle ages.

The sense of generality arises within the mental realm, and is expressed in words, diagrams and symbols in the visible realm. As Shaffer suggested, things on screens and physically present are particular. To appreciate generality you need to move beyond the particular to the general, and this is best activated by words prompting images, supplemented and underpinned by diagrams that are seen as frames from a complex 'film', not single photos. Again, with diagrams, what matters is reading the dimensions-of-possible-variation, paying attention to what is invariant in the midst of change.

For example, the expression $y = 2x + 3$ is an object. It is also the specification of a rule for calculating values of y corresponding to values of x, a function-machine. In the background is a table of values which could be constructed. Associated with the expression is a graph. Most importantly, it is an expression of generality, a specification of a relationship between two 'quantities'. Encountering the graph or the symbols will ideally trigger access to the other and to all the further associations and connections. This all happens in and through mental imagery.

USING IMAGERY TO TEACH MATHEMATICS

If any of the above is at all convincing, then mathematics teaching will be even more effective if we call upon learners to use their natural powers. For I claim that using your own powers is pleasurable; having other people do things for you so you don't use your powers both atrophies those powers and also demeans, disenchants, demoralizes, and disenfranchises. So this is a call for the use of mental imagery in teaching mathematics.

PRE-PARING YOURSELF

Before you go into a lesson, you prepare yourself by deciding on the topic, on the task, on the technical terms and phrases you want learners to begin to use, the techniques they need to become competent in and to gain facility in, and so on. It is useful also to remind yourself of underlying and generative images, contexts, and associations (Griffin & Gates 1988, Mason 2001). In selecting and augmenting the tasks you will give them, you imagine some of the learners and how they will respond. This alerts you to potential difficulties in getting activity going, whether it is a discussion in small groups or whole class, some sort of practical activity, or individual work from worksheets or texts. All of this takes place in your imagination. You can actually enter the class in advance, be there emotionally and cognitively, and anticipate responses.

But you can do more. You can pick something that you would like to do but often forget: perhaps something about how you start or wind-up discussion, perhaps something about your voice tones or the way you pause when you ask a question. Whatever it is, you can make it more likely that you will remember to do it, more likely that the possibility will come to mind in the middle of the lesson, if you spend a few seconds mentally imagining yourself in the class, something happening which might trigger this action you want to take, and yourself taking that action. Push it further: imagine the response from the learners, and imagine yourself responding to that response. Don't just imagine what you want to happen: be realistic and try to foresee unexpected responses.

Of course this is what you always do. But doing it with a little more intention, a little more intensity can make a big difference, can actually assist you to develop your teaching. This approach to working on teaching is elaborated in Mason (2002).

EXPLOITING IMAGERY

One thing you might decide is to spend a few moments in at least one lesson per week, using mental imagery explicitly. Almost any topic has associated with it images, diagrams, and pictures. Most are derived from some material phenomenon, or arise from some mathematical phenomenon which can be imagined or which are used in some imaginable context. It does not have to be totally 'in the mind'. You can use a screen to display mathematical objects such as the sequence above, but drawing attention to the way in which the learners imagine what is coming next. A lesson without the opportunity to generalise is not a mathematics lesson, and mental imagery is the faculty with which we experience generalization.

POST-PARING YOURSELF

'Post-paring' is a made-up word, constructed to signal that pre-paration really has two parts: reflection back on what has happened in the past and imagining forward into the future. 'Reflection' is a vastly over-used term. Here I use it to mean using the power of mental imagery to re-enter salient and significant moments from the recent past in order to learn from them. There is much to say about observing without judging, in order to learn from experience. Suffice it to say here that in order to learn from experience it is at least advantageous, and usually necessary, to reflect actively. This means re-entering moments when you wish, in retrospect, that you had thought of choosing a different action, then imagining yourself making that choice. By being 'back in the moment' it is possible to coordinate cognitive awareness, emotional commitment, and physical presence in a single virtual-state. By linking all three aspects of the psyche together, and by setting up triggers in the form of words which are likely to arise in a situation and which can then provide access to alternative actions coming to mind, you can increase your sensitivity to learners and increase the range of choices available to you at any moment.

Of course this is also what you are trying to support your learners in doing! Real learning consists of increased sensitivity in situations so that possible actions come to mind more readily, thereby offering a range of choices as to what to do to try to resolve problems, whether inside or outside the classroom. By engaging learners in active reflection you can convert mere activity, mere 'doing', into learning. How? Not by exhorting or cajoling; not by mindless repetition; but by getting learners to use their mental imagery.

CONCLUSION

Invoking mental imagery in the teaching of mathematics is not about preparing a few 'imagery lessons' and then reverting to some other practice. It is about being aware of one's own images, and of opportunities to support pupils in working on their imagery. Some learners are stronger at, even have a preference for picturing; others prefer hearing; others still prefer action. Playing to strengths can allows people to use their strength, but developing the all-round thinker by working at non-preferred modes may be more useful in the long run.

If learners are given a diagram from a text, they can be encouraged to work on it and to draw their own. But if diagrams are always provided then they will lack experience in converting technical words into images in order to understand them; if they do not try to describe what they are seeing to others then they will lack experience in speaking from their mathematical images; if they do not have opportunities to invent notations for themselves they will see mathematics as a predetermined catalogue of facts; if they are not exposed to conventional notation they will be unable to make sense of other sources.

I have tried to indicate the central importance of the power to form mental images, which are the means by which we move around and experience the mental world poised between the material world and the world of symbols. It is also the world through which creative energy enters us, and by means of which we express ourselves in the material world. Mental imagery is the means whereby we hold the complexity of a mathematical topic *as* a topic. I have also suggested that the core of mathematics and of mathematical thinking is the expression of generality, and that the world in which that generality is experienced is the world of mental imagery. Consequently effective learning and effective teaching make full use of these natural and far-reaching powers possessed by all learners who come to class. The only question is

Am I making full use of learners' powers,
or am I trying to do some or all of the work for them?

If the latter, then I can expect decline of interest, engagement, and pleasure in and with mathematics. If the former, I can expect increase in interest, engagement, and pleasure in and with mathematics.

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