CHAPTER 3

Chinese postman problem

Learning objectives

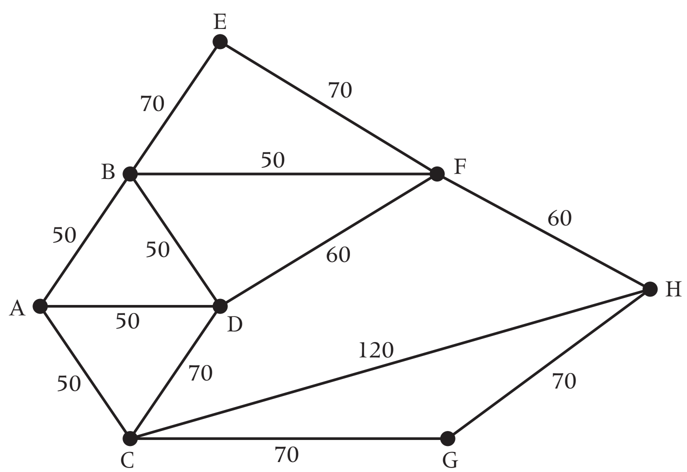
After studying this chapter, you should be able to:

- understand the Chinese postman problem
- apply an algorithm to solve the problem
- understand the importance of the order of vertices of graphs.

3.1 Introduction

In 1962, a Chinese mathematician called Kuan Mei-Ko was interested in a postman delivering mail to a number of streets such that the total distance walked by the postman was as short as possible. How could the postman ensure that the distance walked was a minimum?

In the following example a postman has to start at A, walk along all 13 streets and return to A. The numbers on each edge represent the length, in metres, of each street. The problem is to find a trail that uses all the edges of a graph with minimum length.

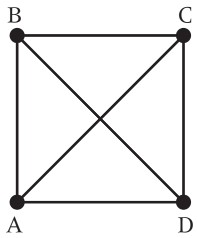


We will return to solving this actual problem later, but initially we will look at drawing various graphs.

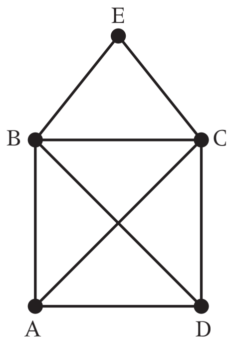
3.2 Traversable graphs

A **traversable** graph is one that can be drawn without taking a pen from the paper and without retracing the same edge. In such a case the graph is said to have an Eulerian trail.

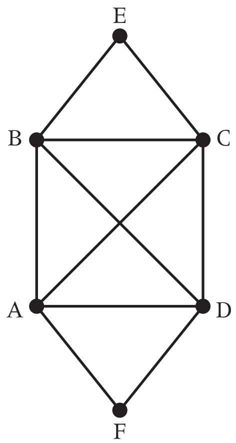
Eulerian trails are dealt with in detail in Chapter 5.



Graph 1



Graph 2



Graph 3

If we try drawing the three graphs shown above we find:

- it is impossible to draw Graph 1 without either taking the pen off the paper or re-tracing an edge
- we can draw Graph 2, but only by starting at either A or D – in each case the path will end at the other vertex of D or A
- Graph 3 can be drawn regardless of the starting position and you will always return to the start vertex.

What is the difference between the three graphs?
In order to establish the differences, we must consider the order of the vertices for each graph. We obtain the following:

Graph 1

Vertex	Order
A	3
B	3
C	3
D	3

Graph 2

Vertex	Order
A	3
B	4
C	4
D	3
E	2

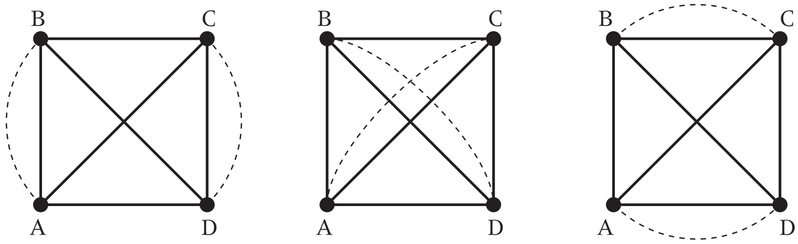
Graph 3

Vertex	Order
A	4
B	4
C	4
D	4
E	2
F	2

When the order of all the vertices is even, the graph is traversable and we can draw it. When there are two odd vertices we can draw the graph but the start and end vertices are different. When there are four odd vertices the graph can't be drawn without repeating an edge.

> An **Eulerian** trail uses all the edges of a graph. For a graph to be Eulerian all the vertices must be of even order.
>
> If a graph has two odd vertices then the graph is said to be **semi-Eulerian**. A trail can be drawn starting at one of the odd vertices and finishing at the other odd vertex.

To draw the graph with odd vertices, edges need to be repeated. To find such a trail we have to make the order of each vertex even. In graph 1 there are four vertices of odd order and we need to pair the vertices together by adding an extra edge to make the order of each vertex four. We can join AB and CD, or AC and BD, or AD and BC.

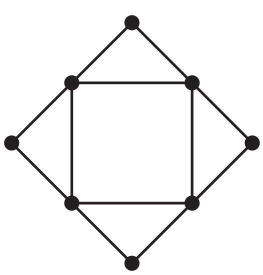


In each case the graph is now traversable.

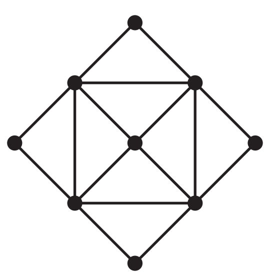
Worked example 3.1

Which of the graphs below is traversable?

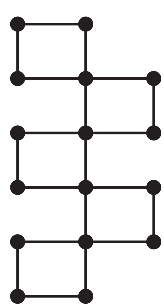
(a)



(b)



(c)



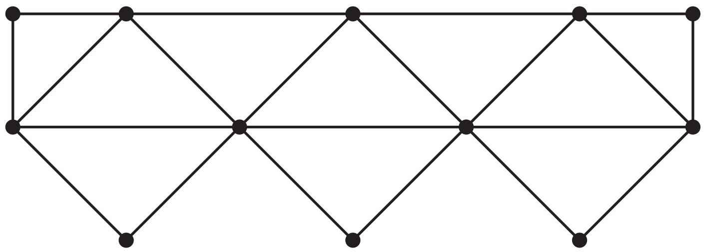
Solution

Graphs **(a)** and **(c)** are traversable as all the vertices are of even order. Graph **(b)** is not traversable as there are vertices of odd order.

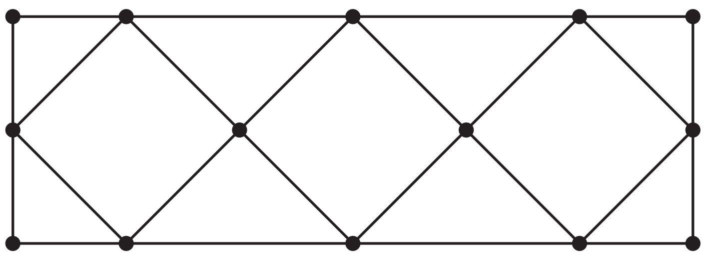
EXERCISE 3A

Which of the graphs below are traversable?

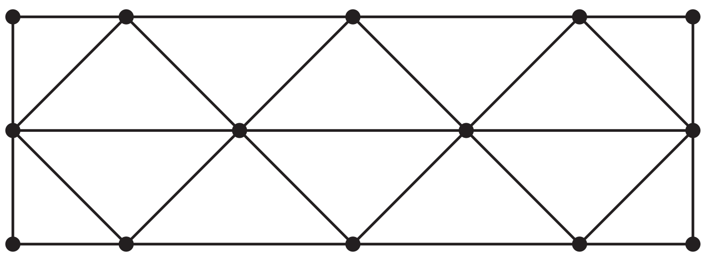
1



2



3



3.3 Pairing odd vertices

If there are two odd vertices there is only one way of pairing them together.
If there are four odd vertices there are three ways of pairing them together.
How many ways are there of pairing six or more odd vertices together?
If there are six odd vertices ABCDEF, then consider the vertex A. It can be paired with any of the other five vertices and still leave four odd vertices. We know that the four odd vertices can be paired in three ways; therefore the number of ways of pairing six odd vertices is $5 \times 3 \times 1 = 15$.

Similarly, if there are eight odd vertices ABCDEFGH, then consider the first odd vertex A. This could be paired with any of the remaining seven vertices and still leave six odd vertices. We know that the six odd vertices can be paired in 15 ways therefore the number of ways of pairing eight odd vertices is $7 \times 5 \times 3 \times 1 = 105$ ways.
We can continue the process in the same way and the results are summarised in the following table.

Number of odd vertices	Number of possible pairings
2	1
4	$3 \times 1 = 3$
6	$5 \times 3 \times 1 = 15$
8	$7 \times 5 \times 3 \times 1 = 105$
10	$9 \times 7 \times 5 \times 3 \times 1 = 945$
n	$(n-1) \times (n-3) \times (n-5) \ldots \times 1$

Exam questions will not be set where candidates will have to pair more than four odd vertices but students do need to be aware of the number of ways of pairing more than four odd vertices.

3.4 Chinese postman algorithm

To find a minimum Chinese postman route we must walk along each edge at least once and in addition we must also walk along the least pairings of odd vertices on one extra occasion.

An algorithm for finding an optimal Chinese postman route is:

Step 1 List all odd vertices.

Step 2 List all possible pairings of odd vertices.

Step 3 For each pairing find the edges that connect the vertices with the minimum weight.

Step 4 Find the pairings such that the sum of the weights is minimised.

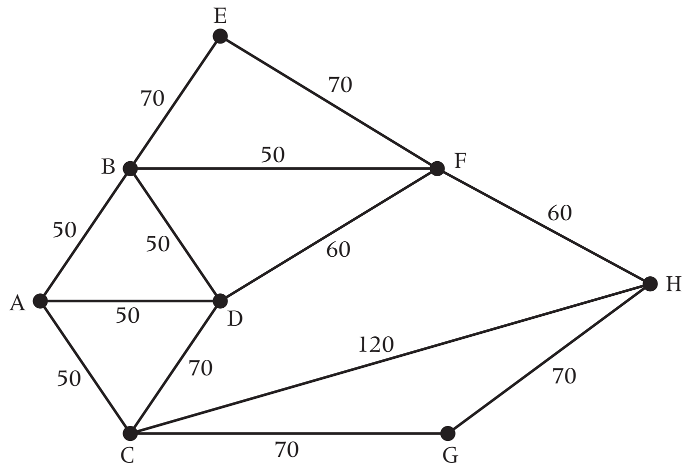
Step 5 On the original graph add the edges that have been found in Step 4.

Step 6 The length of an optimal Chinese postman route is the sum of all the edges added to the total found in Step 4.

Step 7 A route corresponding to this minimum weight can then be easily found.

Worked example 3.2

If we now apply the algorithm to the original problem:

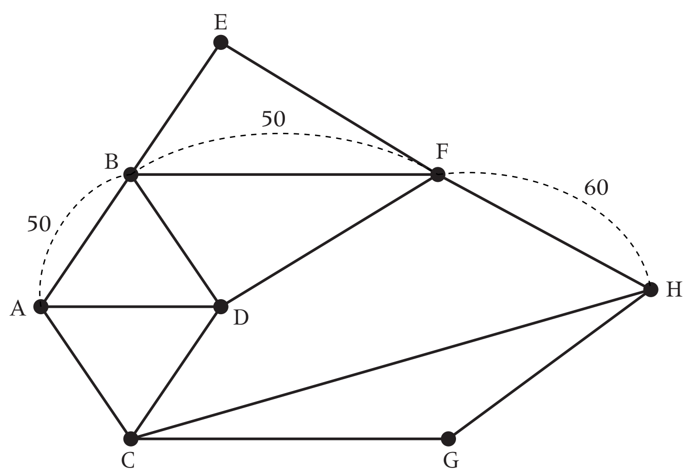


Step 1 The odd vertices are A and H.

Step 2 There is only one way of pairing these odd vertices, namely AH.

Step 3 The shortest way of joining A to H is using the path AB, BF, FH, a total length of 160.

Step 4 Draw these edges onto the original network.



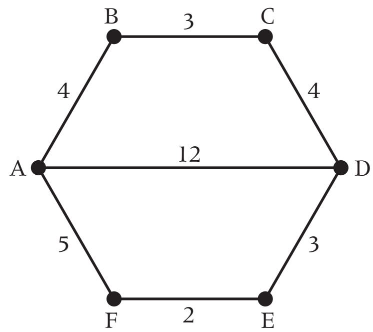
Step 5 The length of the optimal Chinese postman route is the sum of all the edges in the original network, which is 840 m, plus the answer found in Step 4, which is 160 m. Hence the length of the optimal Chinese postman route is 1000 m.

Step 6 One possible route corresponding to this length is ADCGHCABDFBEFHFBA, but many other possible routes of the same minimum length can be found.

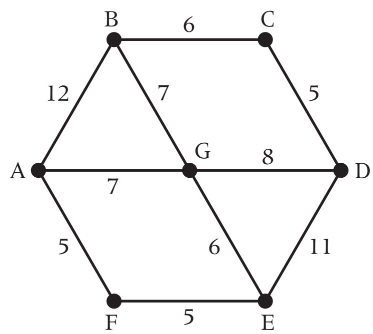
EXERCISE 3B

1 Find the length of an optimal Chinese postman route for the networks below.

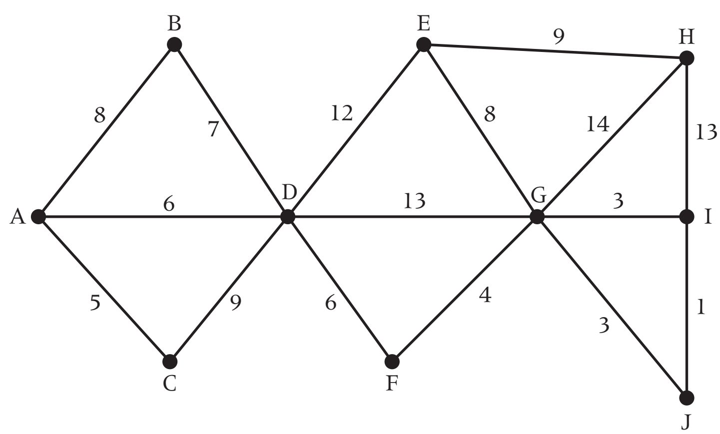
(a)



(b)



(c)



3.5 Finding a route

The method for finding the length of the Chinese postman route is quite straightforward, but to find the list of edges corresponding to this route can be quite tricky, especially in complicated networks. It is useful to calculate how many times each vertex will appear in a Chinese postman route. The following method should be applied before trying to find the route.

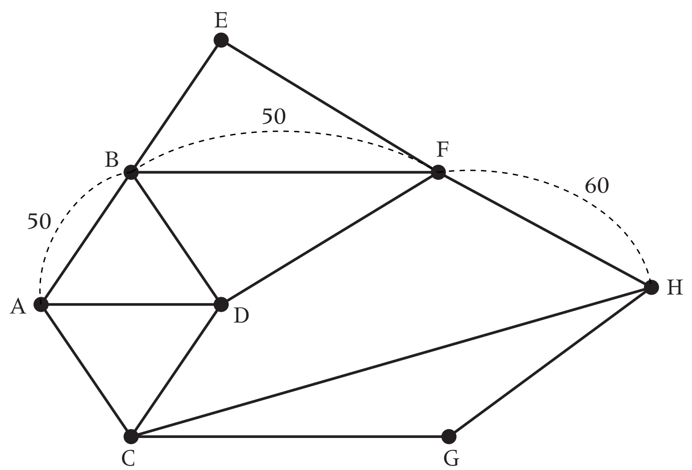
Step 1 On the original diagram add the extra edges to make the graph Eulerian.

Step 2 List the order of each vertex. At this stage each vertex will have an even order.

Step 3 The number of times each edge will appear in a Chinese postman route will be half the order of its vertex, with the exception being vertex A (the start/finish vertex), as this will appear on one extra occasion.

Referring to the diagram below, the orders of the vertices are as follows:

Vertex	**Order**
A	4
B	6
C	4
D	4
E	2
F	6
G	2
H	4



This indicates that the number of times each vertex will appear in the Chinese postman route is:

A $\frac{4}{2} = 2 + 1 = 3$
B $\frac{6}{2} = 3$
C $\frac{4}{2} = 2$
D $\frac{4}{2} = 2$
E $\frac{2}{2} = 1$
F $\frac{6}{2} = 3$
G $\frac{2}{2} = 1$
H $\frac{4}{2} = 2$

The number of vertices in the optimal Chinese postman route is 17. They may be in a different order than in the example above but they must have the number of vertices as indicated in the table.

EXERCISE 3C

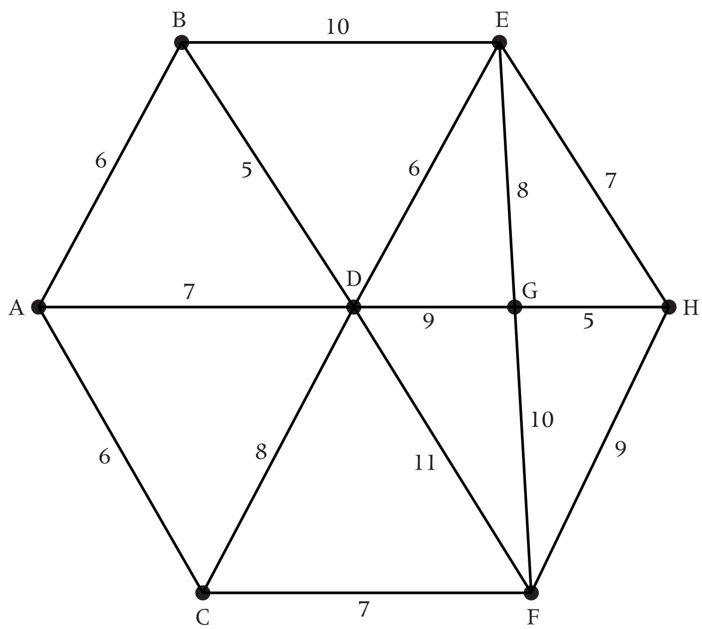
Find a route corresponding to an optimal Chinese postman route for the questions in Exercise 3B.

3.6 Variations of the Chinese postman problem

Occasionally problems may be set where the start vertex and the finish vertex do not have to be the same. Any graph with two odd vertices is semi-Eulerian.
For this type of graph the length of the Chinese postman route is the sum of all the edges of a network.
In a network with four vertices, the graph is semi-Eulerian plus two odd edges. In addition to the start and finish vertices there are two other odd vertices.
The shortest Chinese postman route is the sum of all the edges plus the shortest distance connecting the two remaining odd vertices.

Worked example 3.3

A county council is responsible for maintaining the following network of roads. The number on each edge is the length of the road in miles.



The council office is based at A.

(a) A council worker has to inspect all the roads, starting and finishing at A. Find the length of an optimal Chinese postman route.

(b) A supervisor, based at A, also wishes to inspect all the roads. However, the supervisor lives at H and wishes to start his route at A and finish at H. Find the length of an optimal Chinese postman route for the supervisor.

Solution

(a) There are four odd vertices: A, B, C and H.
There are three ways of pairing these odd vertices and the minimum length of each pairing is:

AB + CH = 6 + 16 = 22
AC + BH = 6 + 17 = 23
AH + BC = 21 + 12 = 33

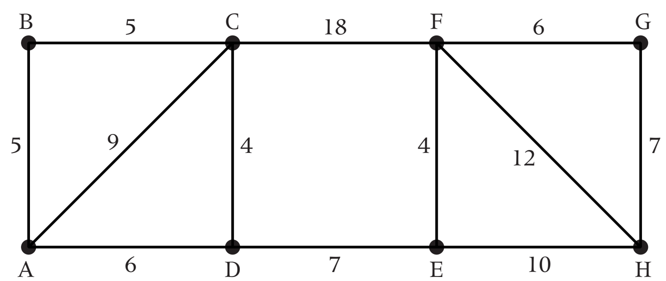
Draw the edges AB and CH onto the network.
The length of all the roads in the network is 116.
The length of an optimal Chinese postman route for the worker is 116 + 22 = 138 miles.

(b) Starting at A and finishing at H leaves two odd vertices B and C.
The minimum distance from B to C is 12.
The length of an optimal Chinese postman route for the supervisor is 116 + 12 = 128 miles.

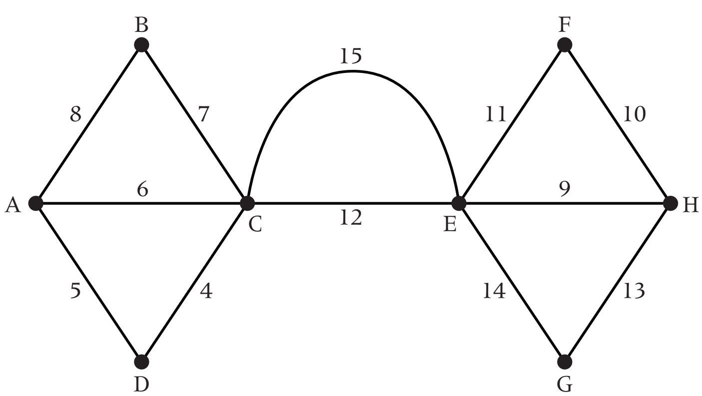
EXERCISE 3D

For each of the networks below find the length of an optimal Chinese postman route starting at A and finishing at H.

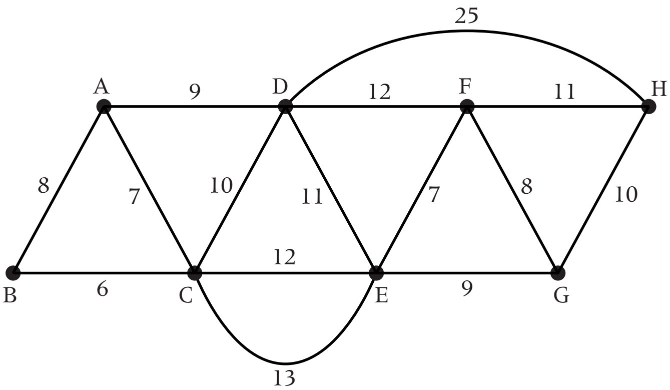
1



2



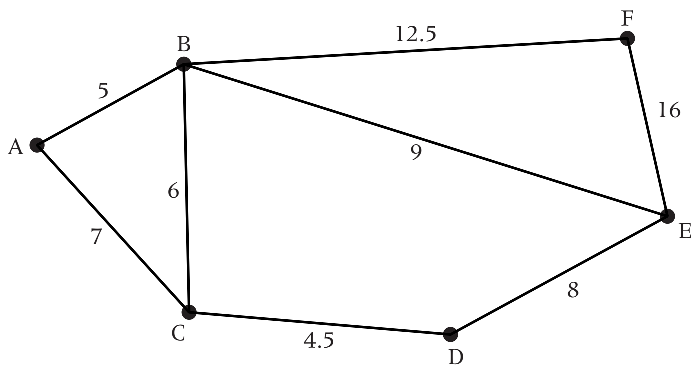
3



MIXED EXERCISE

1 A local council is responsible for gritting roads.

(a) The following diagram shows the lengths of roads, in miles, that have to be gritted.



The gritter is based at A and must travel along all the roads, at least once, before returning to A.

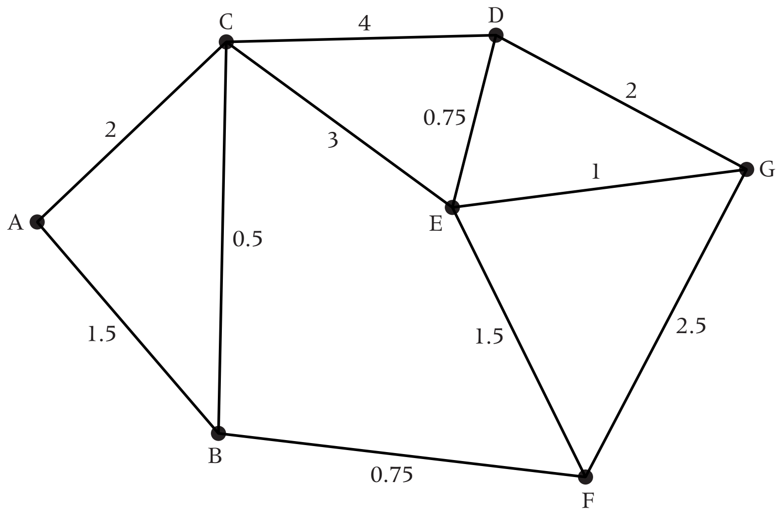
(i) Explain why it is **not** possible to start from A and, by travelling along each road only once, return to A.

(ii) Find an optimal Chinese postman route around the network, starting and finishing at A. State the length of your route.

(b) **(i)** The connected graph of the roads in the area run by another council has six odd vertices. Find the number of ways of pairing these odd vertices.

(ii) For a connected graph with n odd vertices, find an expression for the number of ways of pairing these vertices. [A]

2 A road-gritting service is based at a point A. It is responsible for gritting the network of roads shown in the diagram, where the distances shown are in miles.



(a) Explain why it is **not** possible to start from A and, by travelling along each road only once, return to A.

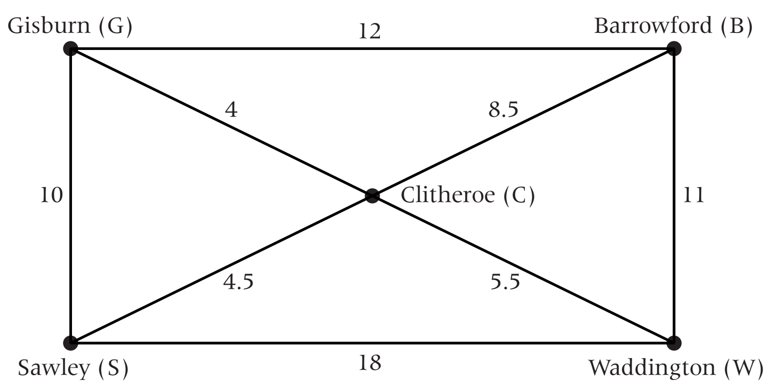
(b) In the network there are four odd vertices, B, D, F and G. List the different ways in which these odd vertices can be arranged as two pairs.

(c) For **each** pairing you have listed in **(b)**, write down the sum of the shortest distance between the first pair and the shortest distance between the second pair.

(d) Hence find an optimal Chinese postman route around the network, starting and finishing at A. State the length of your route. [A]

3 A highways department has to inspect its roads for fallen trees.

(a) The following diagram shows the lengths of the roads, in miles, that have to be inspected in one district.



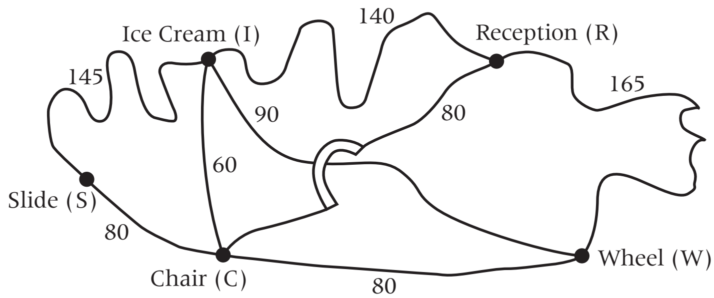
(i) List the three different ways in which the four odd vertices in the diagram can be paired.

(ii) Find the shortest distance that has to be travelled in inspecting all the roads in the district, starting and finishing at the same point.

(b) The connected graph of the roads in another district has six odd vertices. Find the number of ways of pairing these odd vertices.

(c) For a connected graph with n odd vertices, find an expression for the number of ways of pairing these odd vertices. [A]

4 A theme park employs a student to patrol the paths and collect litter. The paths that she has to patrol are shown in the following diagram, where all distances are in metres. The path connecting I and W passes under the bridge which carries the path connecting C and R.



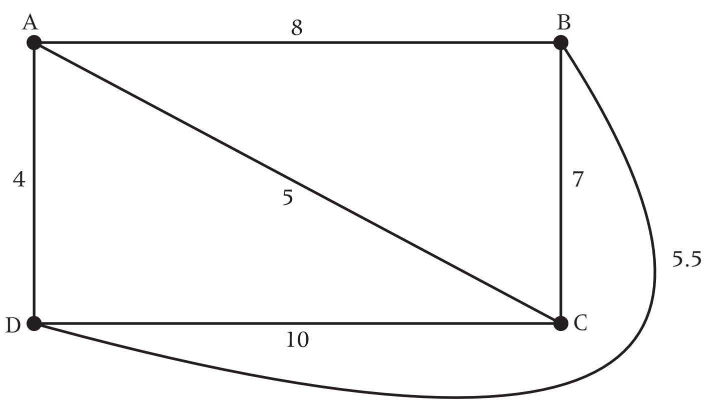
(a) **(i)** Find an optimal Chinese postman route that the student should take if she is to start and finish at Reception (R).

(ii) State the length of your route.

(b) **(i)** A service path is to be constructed. Write down the two places that this path should connect, if the student is to be able to walk along every path without having to walk along any path more than once.

(ii) The distance walked by the student in part **(b)(i)** is shorter than that found in part **(a)(ii)**. Given that the length of the service path is l metres, where l is an integer, find the greatest possible value of l. [A]

5 In the following network the four vertices are odd.

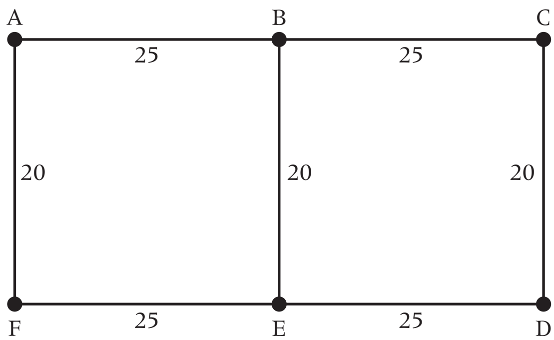


(a) List the different ways in which the vertices can be arranged as two pairs.

(b) For **each** pairing you have listed in **(a)**, write down the sum of the shortest distance between the first pair and the shortest distance between the second pair. Hence find the length of an optimal Chinese postman route around the network.

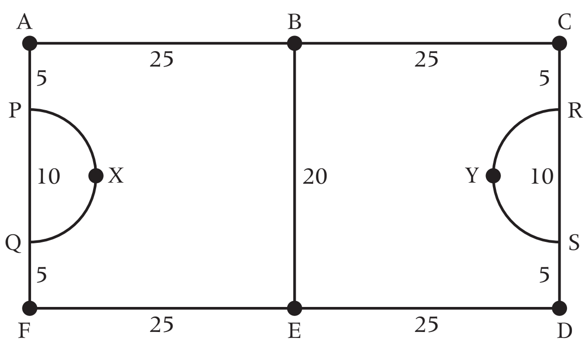
(c) State the minimum number of extra edges that would need to be added to the network to make the network Eulerian. [A]

6 A groundsman at a local sports centre has to mark out the lines of several five-a-side pitches using white paint. He is unsure as to the size of the goal area and he decides to paint the outline as given below, where all the distances are in metres.



(a) He starts and finishes at the point A. Find the minimum total distance that he must walk and give one of the corresponding possible routes.

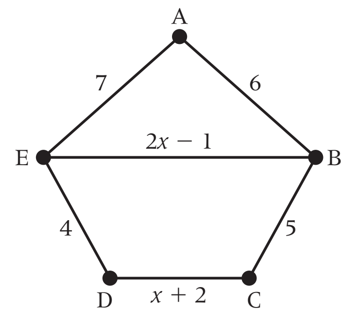
(b) Before he starts to paint the second pitch he is told that each goal area is a semi-circle of radius 5 m, as shown in the diagram below.



(i) He can start at any point but must return to his starting point. State which vertices would be suitable starting points to keep the total distance walked from when he starts to paint the lines until he completes this task to a minimum.

(ii) Find an optimal Chinese postman route around the lines. Calculate the length of your route. [A]

7 The diagram shows a network of roads connecting five villages. The numbers on the roads are the times, in minutes, taken to travel along each road, where $x > 0.5$.



A police patrol car has to travel from its base at B along each road at least once and return to base.

(a) Explain why a route from B to E must be repeated.

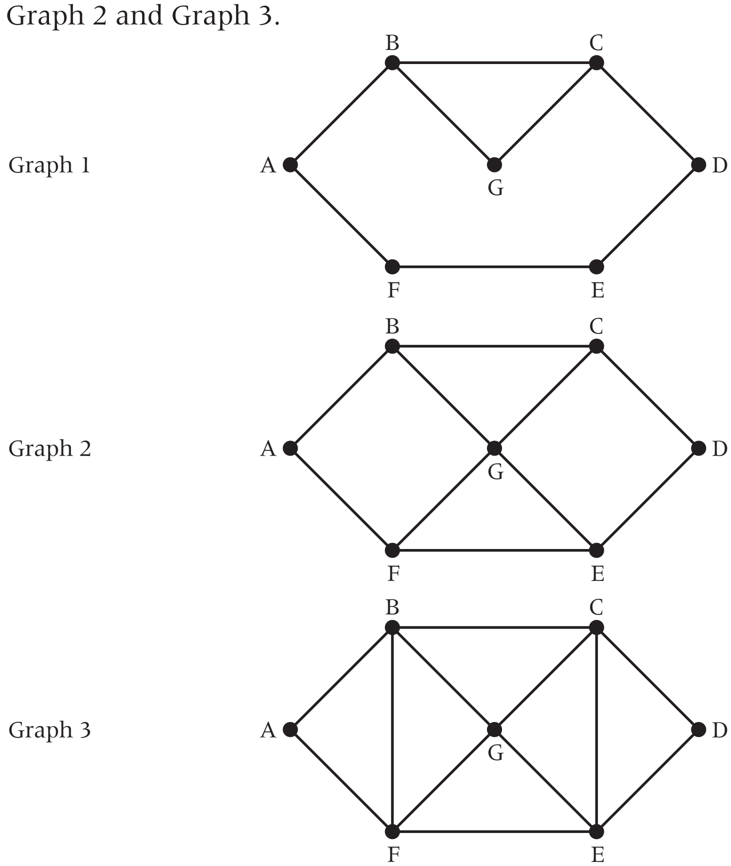
(b) List the routes, and their lengths, from B to E, in terms of x where appropriate.

(c) On a particular day, it is known that $x = 10$.

Find the length of an optimal Chinese postman route on this day. State a possible route corresponding to this minimum length.

(d) Find, no matter what the value of x, which of the three routes should **not** be used if the total length of a Chinese postman route is to be optimal. [A]

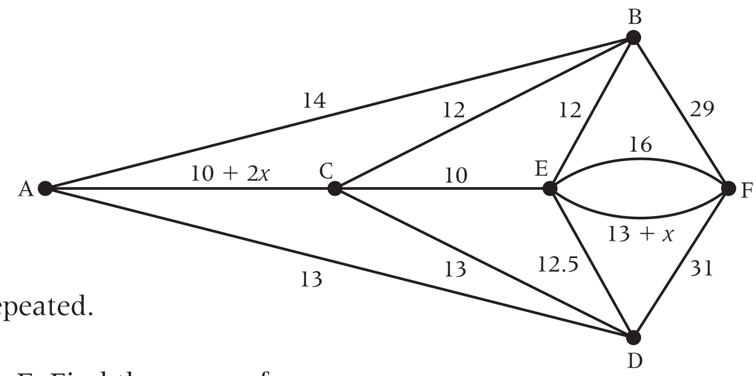
8 The following question refers to the three graphs: Graph 1, Graph 2 and Graph 3.



(a) For **each** of the graphs explain whether or not the graph is Eulerian.

(b) The length of each edge connecting two vertices is 1 unit. Find, for **each** of the graphs, the length of an optimal Chinese postman route, starting and finishing at A. [A]

9 The diagram shows the time, in minutes, for a traffic warden to walk along a network of roads, where $x > 0$.



The traffic warden is to start at A and walk along each road at least once before returning to A.

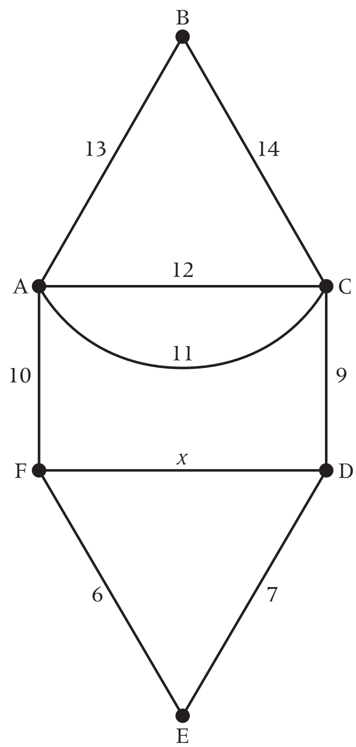
(a) Explain why a section of roads from A to E has to be repeated.

(b) The route ACE is the second shortest route connecting A to E. Find the range of possible values of x.

(c) Find, in terms of x, an expression for the minimum distance that the traffic warden must walk and write down a possible route that he could take.

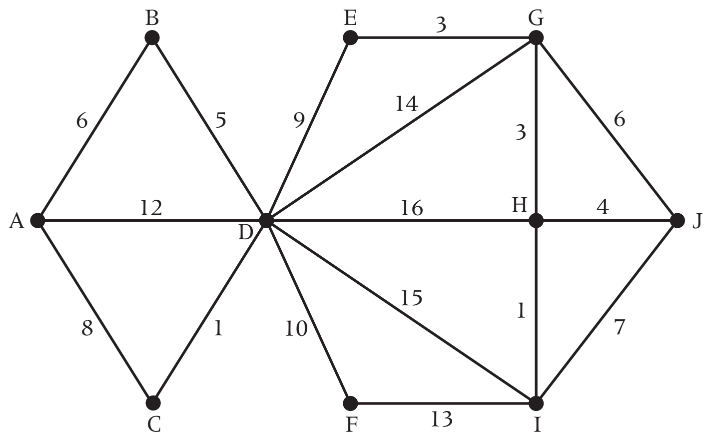
(d) Starting at A, the traffic warden wants to get to F as quickly as possible. Use Dijkstra's algorithm to find, in terms of x, the minimum time for this journey, stating the route that he should take. [A]

10 The following diagram shows a network of roads connecting six towns. The number on each arc represents the distance, in miles, between towns. The road connecting towns D and F has length x miles, where $x < 13$.



An optimal Chinese postman route, starting and finishing at A, has length 100 miles. Find the value of x. [A]

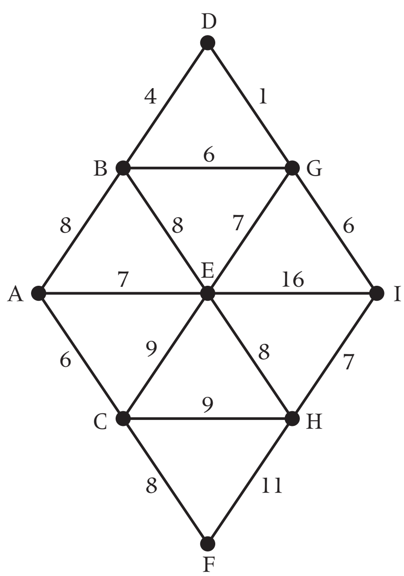
11 The following network shows the distances, in kilometres, of roads connecting ten towns.



(a) An ambulance is based at A and has to respond to an emergency at J. Use Dijkstra's algorithm to find the minimum distance required to travel from A to J, and state the route.

(b) A police motorcyclist, based at town A, has to travel along each of the roads at least once before returning to base at A. Find the minimum total distance the motorcyclist must travel. [A]

12 The following diagram shows the lengths of roads, in miles, connecting nine towns.

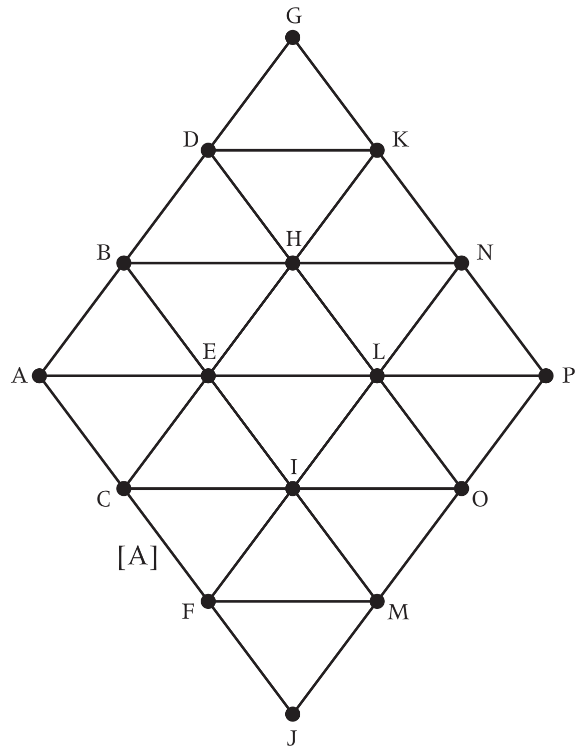


(a) Use Prim's algorithm, starting from A, showing your working at each stage, to find the minimum spanning tree for the network. State its length.

(b) (i) Find an optimal Chinese postman route around the network, starting and finishing at A. You may find the shortest distance between any two towns by inspection.

(ii) State the length of your route. [A]

13 The network on the right has 16 vertices.



(a) Given that the length of each edge is 1 unit, find:

(i) the shortest distance from A to K

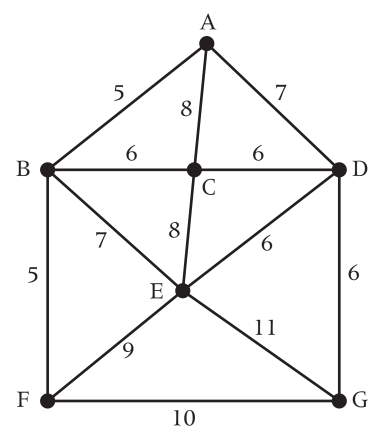
(ii) the length of a minimum spanning tree.

(b) (i) Find the length of an optimal Chinese postman route, starting and finishing at A.

(ii) For such a route, state the edges that would have to be used twice.

(iii) Given that the edges AE and LP are now removed, find the new length of an optimal Chinese postman route, starting and finishing at A. [A]

14 The numbers of parking meters on the roads in a town centre are shown in the network on the right.



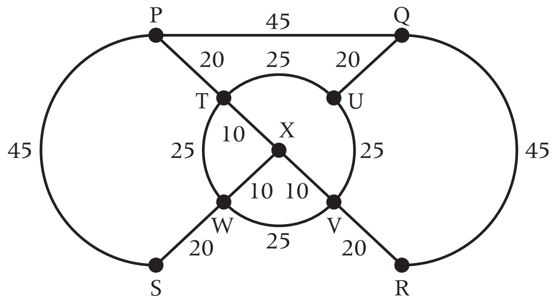
(a) A traffic warden wants to start at A, walk along the roads passing each meter at least once and finish back at A. She wishes to choose her route in order to minimise the number of meters that she passes more than once.

(i) Explain how you know that it will be necessary for her to pass some meters more than once.

(ii) Apply the Chinese postman algorithm to find the minimum number of meters which she will have to pass more than once, and give an example of a suitable route.

(b) At each of the junctions A, B, C, D, E, F and G there is a set of traffic lights. The traffic warden is asked to make a journey, starting and finishing at A, to check that each set of traffic lights is working correctly. Find a suitable route for her which passes 50 or fewer meters. [A]

15 The vertices of the following network represent the chalets in a small holiday park and the arcs represent the paths between them, with the lengths of the paths given in metres.



A gardener wishes to sweep all the paths, starting and finishing at P, and to do so by walking (always on the paths) as short a distance as possible. Apply the Chinese postman algorithm to find the shortest distance the gardener must walk, and give one possible shortest route. [A]

Key point summary

1 A **traversable** graph is one that can be drawn without taking a pen from the paper and without retracing the same edge. In such a case the graph is said to have an Eulerian trail. *p45*

2 An **Eulerian trail** uses all the edges of a graph. For a graph to be Eulerian all the vertices must be of even order. *p46*

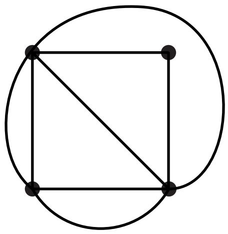
3 If a graph has two odd vertices then the graph is said to be **semi-Eulerian**. A trail can be drawn starting at one of the odd vertices and finishing at the other odd vertex. *p46*

4 A minimum Chinese postman route requires each edge to be walked along at least once and in addition the least pairings of odd vertices must be walked along on one extra occasion. *p48*

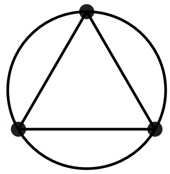
Test yourself	What to review

1 Which of the following networks is traversable? — *Section 3.2*

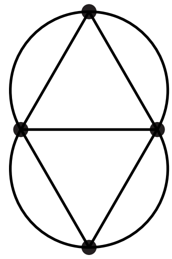
(a)



(b)



(c)



2 Find the number of ways of pairing: — *Section 3.3*

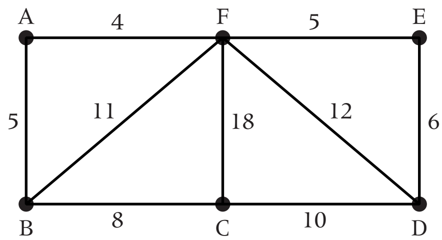
(a) 8 odd vertices,

(b) 12 odd vertices,

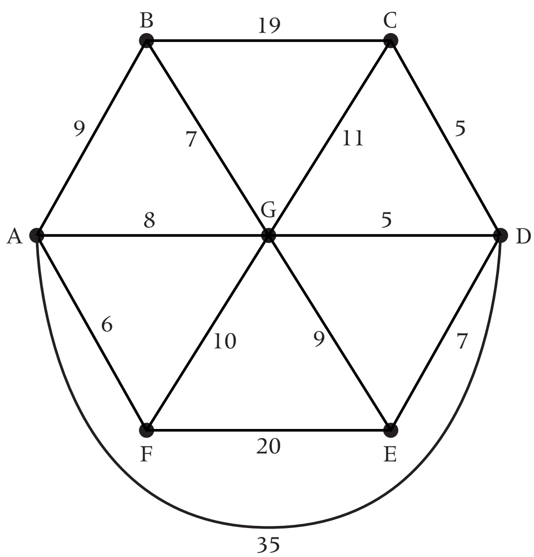
(c) 20 odd vertices.

3 List the ways of pairing the odd vertices in the following networks. For each pairing find the minimum connector. Find the length of an optimal Chinese postman route. Write down one possible route. — *Sections 3.3, 3.4, 3.5*

(a)



(b)



Test yourself ANSWERS

1 **(a)** No **(b)** Yes **(c)** No

2 **(a)** 105 **(b)** 10 395 **(c)** 654 729 075

3 **(a)** BC + FD = 19
BD + FC = 35
BF + DC = 19
Total 79 + 19 = 98
AFEDFEDCFBCBA

(b) BC + EF = 36
BE + CF = 36
BF + CE = 27
Total 151 + 27 = 178
ABCDGCDEGBAGFADEFA